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Algebraic properties of the Dirac oscillator

O L de Lange

Physics Department, University of Natal, PO Box 375, Pietermaritzburg 3200, South Africa

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Abstract. An algebraic (representation-independent) analysis is presented for the Dirac oscillator in an angular momentum basis. The analysis is based on shift operators for energy and angular momentum, and it is similar to that for a non-relativistic isotropic harmonic oscillator. The shift operators generate all the eigenkets of the Dirac oscillator from a 'vacuum' ket. The shift operations yield energy eigenvalues and certain matrix elements. The relationship to the factorization method is discussed.

1. Introduction

Recently there has been considerable interest in the properties and applications of the Dirac oscillator [1-5]. This oscillator is described by a Dirac equation in which the interaction of a particle of rest mass m_0 with an external potential is introduced by the (non-minimal) substitution

$$\boldsymbol{p} \rightarrow \boldsymbol{p} - \mathrm{i} m_0 \omega \beta \boldsymbol{r} \tag{1.1}$$

where ω is a constant and β is a Dirac matrix [1, 6]. The energy spectrum for the Dirac oscillator can be determined analytically [7] and the degeneracies have been discussed in terms of a hidden supersymmetry [3, 8, 9].

It is known that in certain aspects the Dirac oscillator is related to an isotropic three-dimensional non-relativistic harmonic oscillator (hereafter referred to as an ordinary oscillator). For example, in the non-relativistic limit the equation satisfied by the large components of the wavefunction for the Dirac oscillator corresponds to the wave equation of an ordinary oscillator with spin-orbit coupling [1]. Quesne and Moshinsky [4] have used this property to determine the symmetry Lie algebra of the Dirac oscillator. In their work the generators of the symmetry algebra are constructed using properties of the non-relativistic coordinate-space wavefunctions.

The purpose of this paper is threefold. Firstly we show that the Dirac oscillator can be treated by a purely algebraic method, that is without choosing any representation space. This algebraic analysis is based on abstract shift operators that generate all the eigenkets of the oscillator from a given eigenket (such as the 'vacuum' $|0\rangle$). Secondly we show that the analysis can be performed from first principles in a simple manner which is similar to that for an ordinary oscillator in an angular momentum basis [10, 11]. Thirdly we discuss the properties of the above shift operators.

In section 2 we discuss commuting operators for the Dirac oscillator and in section 3 we obtain shift operators for the eigenvalues of these commuting operators. Algebraic properties of the shift operators and the relationship to the factorization method are considered in section 4. In section 5 matrix elements, phase factors, and wavefunctions are discussed.

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The Dirac oscillator is one of the few relativistic spherically symmetric problems for which such a complete algebraic treatment is possible†. In addition to their intrinsic interest the results presented here could be applied, for example, in the construction of coherent angular momentum states‡.

2. Commuting operators for the Dirac oscillator

The equation for the Dirac oscillator is (we set $\hbar = c = 1$)

$$H|\Psi\rangle = -\mathrm{i}\frac{\partial}{\partial t}|\Psi\rangle \tag{2.1}$$

where

$$H = \boldsymbol{\alpha}\cdot(\boldsymbol{p} - \mathrm{i}m_0\omega\beta\boldsymbol{r}) + m_0\beta. \tag{2.2}$$

For the Dirac matrices we choose

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{2.3}$$

and

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

where σ_i are the Pauli spin matrices. For the total, orbital, and spin angular momentum operators we adopt the usual notation

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{2.5}$$

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{2.6}$$

$$\boldsymbol{S} = \tfrac{1}{2}\boldsymbol{\Sigma} \tag{2.7}$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{2.8}$$

It is straightforward to show that H, $\boldsymbol{J}^2$, J_z, $\boldsymbol{S}^2$, and

$$K = \beta(\boldsymbol{\Sigma}\cdot\boldsymbol{L} + 1) \tag{2.9}$$

are a set of commuting operators. We consider simultaneous eigenkets of these operators and write

$$H|\Psi\rangle = E|\Psi\rangle \tag{2.10}$$

$$\boldsymbol{J}^2|\Psi\rangle = j(j+1)|\Psi\rangle \tag{2.11}$$

$$J_z|\Psi\rangle = m|\Psi\rangle \tag{2.12}$$

$$\boldsymbol{S}^2|\Psi\rangle = \tfrac{3}{4}|\Psi\rangle \tag{2.13}$$

$$K|\Psi\rangle = k|\Psi\rangle. \tag{2.14}$$

Here $m = -j, -j+1, \ldots, j$ and the values of E, j, and k are to be determined.

† The Dirac-Coulomb problem can also be treated in this manner, although the analysis is complicated (see [12]).

‡ Coherent angular momentum states of the ordinary oscillator have been studied by Bracken and Leemon [13].

For our purposes it is helpful to express the eigenvalue equations in 2×2 block form. Thus we write

$$|\Psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix} \exp(-\mathrm{i}Et). \tag{2.15}$$

Then (2.1) yields the coupled equations

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p} - \mathrm{i}m_0\omega\boldsymbol{\sigma}\cdot\boldsymbol{r})|\phi\rangle = (E+m_0)|\chi\rangle \tag{2.16}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p} + \mathrm{i}m_0\omega\boldsymbol{\sigma}\cdot\boldsymbol{r})|\chi\rangle = (E-m_0)|\phi\rangle. \tag{2.17}$$

Decoupling these equations we obtain

$$\mathscr{H}|\phi\rangle = \omega\left(\frac{E^2 - m_0^2}{2m_0\omega} + \mathscr{K} + \tfrac{1}{2}\right)|\phi\rangle \tag{2.18}$$

and

$$\mathscr{H}|\chi\rangle = \omega\left(\frac{E^2 - m_0^2}{2m_0\omega} - \mathscr{K} - \tfrac{1}{2}\right)|\chi\rangle \tag{2.19}$$

where

$$\mathscr{H} = \frac{1}{2m_0}\boldsymbol{p}^2 + \tfrac{1}{2}m_0\omega^2\boldsymbol{r}^2 \tag{2.20}$$

and

$$\mathscr{K} = \boldsymbol{\sigma}\cdot\boldsymbol{L} + 1. \tag{2.21}$$

Equations (2.14), (2.15), (2.9), and (2.4) show that

$$\mathscr{K}|\phi\rangle = k|\phi\rangle \tag{2.22}$$

$$\mathscr{K}|\chi\rangle = -k|\chi\rangle. \tag{2.23}$$

From (2.11) and (2.15), $|\phi\rangle$ and $|\chi\rangle$ are eigenkets of $\tilde{\boldsymbol{J}}^2$ where

$$\tilde{\boldsymbol{J}} = \boldsymbol{L} + \tfrac{1}{2}\boldsymbol{\sigma}. \tag{2.24}$$

The corresponding eigenvalues are $j(j+1)$. This result and (2.22) shows that $|\phi\rangle$ is an eigenket of $\boldsymbol{L}^2$ with eigenvalues $l(l+1)$, and thus the theory of combination of two angular momenta requires

$$j = l \pm \tfrac{1}{2}. \tag{2.25}$$

Similarly, $|\chi\rangle$ is an eigenket of $\boldsymbol{L}^2$ with eigenvalues $l'(l'+1)$, where $l' = l \pm 1$ for $j = l \pm \frac{1}{2}$. Hence the Dirac quantum number in (2.22) and (2.23) has the values

$$k = -\eta(j + \tfrac{1}{2}) \tag{2.26}$$

where

$$\eta = (-1)^{j+l+1/2}. \tag{2.27}$$

The Hamiltonian (2.2) commutes with the parity operator and (2.15) is an eigenket of parity with eigenvalue $(-1)^l$.

We can summarize the eigenvalue equations for $|\phi\rangle$ and $|\chi\rangle$ in the convenient form

$$\mathscr{H}|njmq\rangle = \omega(n+1+\delta)|njmq\rangle \tag{2.28}$$

$$\tilde{J}^2|njmq\rangle = j(j+1)|njmq\rangle \tag{2.29}$$

$$\tilde{J}_z|njmq\rangle = m|njmq\rangle \tag{2.30}$$

$$L^2|njmq\rangle = q(q+1)|njmq\rangle. \tag{2.31}$$

Here $|njmq\rangle$ denotes either $|\phi\rangle$ or $|\chi\rangle$. If $|njmq\rangle = |\phi\rangle$,

$$\begin{aligned} \delta &= \tfrac{1}{2} \\ q &= l. \end{aligned} \tag{2.32}$$

If $|njmq\rangle = |\chi\rangle$,

$$\begin{aligned} \delta &= -\tfrac{1}{2} \\ q &= l - \eta. \end{aligned} \tag{2.33}$$

The dimensionless parameter n is defined as

$$n = (2m_0\omega)^{-1}(E^2 - m_0^2) + k - 1. \tag{2.34}$$

From (2.34) and (2.26) the positive and negative energies are given in terms of n and j by

$$E_{nj} = \pm m_0\{1 + 2\omega m_0^{-1}[n + 1 + \eta(j+\tfrac{1}{2})]\}^{1/2}. \tag{2.35}$$

The operators $\mathscr{H}$, $\tilde{J}^2$, $\tilde{J}_z$, and L^2 are a set of commuting operators for the elements $|\phi\rangle$ and $|\chi\rangle$ of the Dirac oscillator (see section 5). In the next section we determine shift operators for the quantum numbers n, j, m, and q. In these calculations we need the normalization conditions for $|\phi\rangle$ and $|\chi\rangle$. The normalization $\langle\Psi|\Psi\rangle = 1$ of the Dirac ket (2.15) requires

$$\langle\phi|\phi\rangle + \langle\chi|\chi\rangle = 1. \tag{2.36}$$

Multiplying (2.16) and (2.17) on the left by $\langle\chi|$ and $\langle\phi|$, respectively, and using the Hermitian properties of $\boldsymbol{r}$, $\boldsymbol{p}$, and $\boldsymbol{\sigma}$, we have

$$(E + m_0)\langle\chi|\chi\rangle = (E - m_0)\langle\phi|\phi\rangle. \tag{2.37}$$

From (2.36) and (2.37) we see that the normalization conditions for $|\phi\rangle$ and $|\chi\rangle$ are

$$\langle\phi|\phi\rangle = \frac{1}{2}\left(1 + \frac{m_0}{E}\right) \tag{2.38}$$

$$\langle\chi|\chi\rangle = \frac{1}{2}\left(1 - \frac{m_0}{E}\right). \tag{2.39}$$

In the non-relativistic limit the norm of $|\chi\rangle$ ($|\phi\rangle$) is small for positive (negative) energy solutions.

3. Shift operators

The operator $\mathscr{H}$ in (2.28) is the same as the Hamiltonian for an ordinary oscillator. Thus to construct shift operators for the eigenkets $|njmq\rangle$ we use the same procedure as for the ordinary oscillator in an angular momentum basis. Following [10] and [11] we adopt the ansatz

$$\boldsymbol{D}(\omega)=\boldsymbol{a}^{\dagger}\times\boldsymbol{L}+\mathrm{i}\boldsymbol{a}^{\dagger}\mathscr{S} \tag{3.1}$$

where

$$\boldsymbol{a}^{\dagger}=(2m_0|\omega|)^{-1/2}(-\mathrm{i}\boldsymbol{p}+m_0\omega\boldsymbol{r}) \tag{3.2}$$

is the boson creation operator and $\mathscr{S}$ is to be determined.

If

$$[\mathscr{H},\mathscr{S}]=0 \tag{3.3}$$

then

$$[\mathscr{H},\boldsymbol{D}(\omega)]=\omega\boldsymbol{D}(\omega). \tag{3.4}$$

If

$$[\boldsymbol{L}^2,\mathscr{S}]=0 \tag{3.5}$$

and

$$\mathscr{S}^2+\mathscr{S}-\boldsymbol{L}^2=0 \tag{3.6}$$

then

$$[\boldsymbol{L}^2,\boldsymbol{D}(\omega)]=-2\boldsymbol{D}(\omega)\mathscr{S}. \tag{3.7}$$

The solution

$$\mathscr{S}=\boldsymbol{\sigma}\cdot\boldsymbol{L} \tag{3.8}$$

to (3.6) satisfies (3.3) and (3.5).

In the rest of this paper we suppose that $\mathscr{S}$ is given by (3.8); thus

$$\boldsymbol{D}(\omega)=\boldsymbol{a}^{\dagger}\times\boldsymbol{L}+\mathrm{i}\boldsymbol{a}^{\dagger}(\boldsymbol{\sigma}\cdot\boldsymbol{L}). \tag{3.9}$$

Then

$$\boldsymbol{D}(-\omega)=-\boldsymbol{a}\times\boldsymbol{L}-\mathrm{i}\boldsymbol{a}(\boldsymbol{\sigma}\cdot\boldsymbol{L}) \tag{3.10}$$

where

$$\boldsymbol{a}=(2m_0|\omega|)^{-1/2}(\mathrm{i}\boldsymbol{p}+m_0\omega\boldsymbol{r}) \tag{3.11}$$

is the boson annihilation operator.

Because $\tilde{\boldsymbol{J}}$ commutes with (3.8) we see that $\boldsymbol{D}(\omega)$ is a vector operator with respect to $\tilde{\boldsymbol{J}}$

$$[\tilde{J}_i, D_j(\omega)]=\mathrm{i}\varepsilon_{ijk}D_k(\omega). \tag{3.12}$$

It is also straightforward to show that

$$[\mathscr{K},\boldsymbol{D}(\omega)]=-D(\omega). \tag{3.13}$$

From the commutators (3.4), (3.7), (3.12), (3.13) and (2.22), (2.23), (2.26)-(2.31) we obtain the twelve shift operations

$$D_\mu(\pm\omega)|njmq\rangle_{\mathrm{U}}=\left(\frac{1+m_0/E_{nj}}{1+m_0/E_{n\pm 1,j+\eta}}\right)^{1/2}\alpha_\mu^{\pm}(njm)|n\pm 1, j+\eta, m+\mu, q+\eta\rangle_{\mathrm{U}} \tag{3.14}$$

and

$$D_\mu(\pm\omega)|njmq\rangle_{\mathrm{L}}=\left(\frac{1-m_0/E_{nj}}{1-m_0/E_{n\pm 1,j-\eta}}\right)^{1/2}\beta_\mu^{\pm}(njm)|n\pm 1, j-\eta, m+\mu, q-\eta\rangle_{\mathrm{L}}. \tag{3.15}$$

Here $\mu=0$ or ± 1 and

$$D_0=D_z \qquad D_{\pm 1}=D_x\pm \mathrm{i}D_y. \tag{3.16}$$

The coefficients $\alpha_\mu^\pm$ and $\beta_\mu^\pm$ are calculated below. The energy-dependent factors in (3.14) and (3.15) have been included to take account of the normalization conditions (2.38) and (2.39); in these factors E_{nj} is given by (2.35) and (3.33). The subscripts U and L in (3.14) and (3.15) indicate whether $|njmq\rangle$ is the upper or lower ket ($|\phi\rangle$ or $|\chi\rangle$) in (2.15).

Using (3.14), (3.15), and the orthogonality of the kets we obtain an additional twelve shift operations

$$D_\mu^\dagger(\pm\omega)|njmq\rangle_{\mathrm{U}} =\left(\frac{1+m_0/E_{nj}}{1+m_0/E_{n\mp 1,j-\eta}}\right)^{1/2} \times[\alpha_{-\mu}^{\pm}(n\mp 1, j-\eta, m+\mu)]^*|n\mp 1, j-\eta, m+\mu, q-\eta\rangle_{\mathrm{U}} \tag{3.17}$$

and

$$D_\mu^\dagger(\pm\omega)|njmq\rangle_{\mathrm{L}} =\left(\frac{1-m_0/E_{nj}}{1-m_0/E_{n\mp 1,j+\eta}}\right)^{1/2} \times[\beta_{-\mu}^{\pm}(n\mp 1, j+\eta, m+\mu)]^*|n\mp 1, j+\eta, m+\mu, q+\eta\rangle_{\mathrm{L}}. \tag{3.18}$$

Here

$$D_0^\dagger=D_z^\dagger \qquad D_{\pm 1}^\dagger=D_x^\dagger\pm \mathrm{i}D_y^\dagger. \tag{3.19}$$

Explicit expressions for $\boldsymbol{D}^\dagger$ are given below (see section 4).

Next we determine the coefficients $\alpha_\mu^\pm$ and $\beta_\mu^\pm$ in the above shift operations. To calculate the magnitudes of these coefficients we use the identities (4.1), (4.11), (4.13), and (4.15). Then

$$[D_\mu(\pm\omega)]^\dagger D_\mu(\pm\omega)=(2m_0\omega)^{-1}R^\dagger(\pm\omega)[(\boldsymbol{\sigma}\times\boldsymbol{L})_\mu]^\dagger(\boldsymbol{\sigma}\times\boldsymbol{L})_\mu R(\pm\omega) \tag{3.20}$$

because $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})^2=1$. Substituting (4.18) and (4.19) in (3.20) and noting that the scalar operators $R(\pm\omega)$ commute with the right-hand sides of (4.18) and (4.19) we have

$$[D_0(\pm\omega)]^\dagger D_0(\pm\omega)=(\omega^{-1}\mathscr{H}\pm\tfrac{1}{2}\mp\boldsymbol{\sigma}\cdot\boldsymbol{L})(\boldsymbol{L}^2-\tilde{J}_z^2+\tfrac{1}{4}) \tag{3.21}$$

and

$$[D_\mu(\pm\omega)]^\dagger D_\mu(\pm\omega)=(\omega^{-1}\mathscr{H}\pm\tfrac{1}{2}\mp\boldsymbol{\sigma}\cdot\boldsymbol{L})[(\boldsymbol{\sigma}\cdot\boldsymbol{L}-\mu\tilde{J}_z)^2-\tfrac{1}{4}] \tag{3.22}$$

where $\mu = \pm 1$. The expectation values of the right-hand sides of (3.21) and (3.22) can be written down using the eigenvalue equations (2.22), (2.23), and (2.28)-(2.31). Thus we obtain

$$\alpha_\mu^\pm(njm) = \theta_\mu^\pm(n \pm \eta j \pm \tfrac{1}{2}\eta \pm \tfrac{3}{2} + \tfrac{3}{2})\gamma_\mu(jm\eta) \tag{3.23}$$

$$\beta_\mu^\pm(njm) = \phi_\mu^\pm(n \mp \eta j \mp \tfrac{1}{2}\eta \pm \tfrac{3}{2} + \tfrac{1}{2})\gamma_\mu(j, m, -\eta) \tag{3.24}$$

where $\theta_\mu^\pm$ and $\phi_\mu^\pm$ are phase factors (see section 5) and

$$\gamma_0(jm\eta) = [(j + \tfrac{1}{2}\eta + \tfrac{1}{2})^2 - m^2]^{1/2} \tag{3.25}$$

$$\gamma_{\pm 1}(jm\eta) = [(\eta j + \tfrac{1}{2}\eta + 1 \pm m)^2 - \tfrac{1}{4}]^{1/2}. \tag{3.26}$$

The above shift operations change both the energy and the angular momentum. It is also useful to construct shift operators that change only the energy; according to (3.14), (3.15), (3.17), and (3.18) these are given by

$$[D_\mu(\mp\omega)]^\dagger D_\mu(\pm\omega) \qquad (\mu = 0, \pm 1).$$

Using the factorizations (4.1), (4.11), (4.18), and (4.19) it is straightforward to show that

$$[D_0(\mp\omega)]^\dagger D_0(\pm\omega) = Q^\pm(L^2 - \tilde{J}_z^2 + \tfrac{1}{4}) \tag{3.27}$$

$$[D_\mu(\mp\omega)]^\dagger D_\mu(\pm\omega) = Q^\pm[(\boldsymbol{\sigma}\cdot\boldsymbol{L} \mp \tilde{J}_z)^2 - \tfrac{1}{4}] \qquad (\mu = \pm 1) \tag{3.28}$$

where

$$Q^\pm = \pm \mathrm{i}\boldsymbol{r}\cdot\boldsymbol{p} - m_0\omega r^2 + \omega^{-1}\mathscr{H} \pm \tfrac{3}{2} \tag{3.29}$$

$$= \pm \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r} + (m_0\omega)^{-1}p^2 - \omega^{-1}\mathscr{H} \mp \tfrac{3}{2}. \tag{3.30}$$

Using (3.27), (2.30), (2.31) and the shift operations involving $D_0(\pm\omega)$ and $D_0^\dagger(\pm\omega)$ we find

$$Q^\pm|njmq\rangle_\mathrm{U} = \eta\left(\frac{1 + m_0/E_{nj}}{1 + m_0/E_{n\pm 2,j}}\right)^{1/2} \times [(n - \eta j - \tfrac{1}{2}\eta + 1 \pm 1)(n + \eta j + \tfrac{1}{2}\eta + 2 \pm 1)]^{1/2}|n \pm 2, j, m, q\rangle_\mathrm{U} \tag{3.31}$$

and

$$Q^\pm|njmq\rangle_\mathrm{L} = \eta\left(\frac{1 - m_0/E_{nj}}{1 - m_0/E_{n\pm 2,j}}\right)^{1/2} \times [(n - \eta j - \tfrac{1}{2}\eta + 1 \pm 1)(n + \eta j + \tfrac{1}{2}\eta \pm 1)]^{1/2}|n \pm 2, j, m, q\rangle_\mathrm{L}. \tag{3.32}$$

In (3.31) and (3.32) we have set the phase factors equal to η (see [5] and section 5).

Non-negativity of the norm in (3.31) and (3.32) requires that the lowering operations must terminate at a minimum value $n' = j + \tfrac{1}{2}\eta$, except that for $j = l + \tfrac{1}{2}$, $n' = j + \tfrac{3}{2}$ in $|njmq\rangle_\mathrm{L}$. The raising operations in (3.31) and (3.32) do not terminate, and if $|n'jmq\rangle$ is normalizable then so are the kets with

$$n = 2N + j + \tfrac{1}{2}\eta \tag{3.33}$$

where

$$N = 0, 1, 2, \ldots . \tag{3.34}$$

For $j = l + \tfrac{1}{2}$, $|njmq\rangle_\mathrm{L}$ does not exist if $N = 0$ (that is, $n = j - \tfrac{1}{2}$). Equations (2.35) and (3.33) yield the well known energy eigenvalues of the Dirac oscillator [1, 3, 7].

From a normalized ket $|njmq\rangle_{\mathrm{U}}$ (or $|njmq\rangle_{\mathrm{L}}$) the shift operators presented above and the coupled equations (2.16) and (2.17) will generate all the kets of the Dirac oscillator. The above analysis for the Dirac oscillator is simpler than that for the ordinary oscillator [10, 11]. This is due to use of the operator (3.8) instead of the number operator $(\boldsymbol{L}^2+\frac{1}{4})^{1/2}$ that occurs in the non-relativistic problem [10, 11]. This situation is reminiscent of the fact that it is easier to treat the Coulomb problem for a Pauli particle than it is for a non-relativistic particle [14].

4. Factorizations

In this section we discuss factorizations of the shift operators $\boldsymbol{D}(\pm\omega)$ and $\boldsymbol{D}^\dagger(\pm\omega)$. These factorizations are useful in applications of the shift operators and in establishing the relationship with the factorization method for constructing shift operators [15].

The vector operators $\boldsymbol{D}(\pm\omega)$ can be factorized into a product of a vector operator and a scalar operator, and this factorization can be performed in two different ways. The first such factorization is

$$\boldsymbol{D}(\pm\omega)=(2m_0\omega)^{-1/2}\boldsymbol{U}R(\pm\omega) \tag{4.1}$$

where

$$\boldsymbol{U}=\hat{\boldsymbol{r}}\times\boldsymbol{L}+\mathrm{i}\hat{\boldsymbol{r}}(\boldsymbol{\sigma}\cdot\boldsymbol{L}) \tag{4.2}$$

and

$$R(\omega)=-\mathrm{i}p_r-r^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{L}+m_0\omega r. \tag{4.3}$$

Here p_r is the usual radial momentum operator

$$p_r=r^{-1}(\boldsymbol{r}\cdot\boldsymbol{p}-\mathrm{i}\hbar). \tag{4.4}$$

It is Hermitian with respect to $|njmq\rangle$ (see section 5) and it satisfies the commutation relation

$$[p_r,f(r)]=-\mathrm{i}\hbar\frac{\mathrm{d}f}{\mathrm{d}r}. \tag{4.5}$$

Thus p_r is the canonical conjugate of r.

The second factorization is

$$\boldsymbol{D}(\pm\omega)=-\mathrm{i}(2m_0\omega)^{-1/2}\boldsymbol{V}P(\pm\omega) \tag{4.6}$$

where

$$\boldsymbol{V}=\hat{\boldsymbol{p}}\times\boldsymbol{L}+\mathrm{i}\hat{\boldsymbol{p}}(\boldsymbol{\sigma}\cdot\boldsymbol{L}) \tag{4.7}$$

and

$$P(\omega)=\mathrm{i}m_0\omega r_p-m_0\omega p^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{L}+p. \tag{4.8}$$

Here

$$r_p=p^{-1}(\boldsymbol{p}\cdot\boldsymbol{r}+\mathrm{i}\hbar) \tag{4.9}$$

is the canonical conjugate of p; it is Hermitian with respect to $|njmq\rangle$ (see section 5) and it satisfies

$$[r_p,f(p)]=\mathrm{i}\hbar\frac{\mathrm{d}f}{\mathrm{d}p}. \tag{4.10}$$

The factorizations of the adjoints $\boldsymbol{D}^{\dagger}(\pm\omega)$ are

$$\boldsymbol{D}^{\dagger}(\pm\omega)=(2m_0\omega)^{-1/2}R^{\dagger}(\pm\omega)\boldsymbol{U}^{\dagger} \tag{4.11}$$

$$=\mathrm{i}(2m_0\omega)^{-1/2}P^{\dagger}(\pm\omega)\boldsymbol{V}^{\dagger}. \tag{4.12}$$

The vector operators $\boldsymbol{U}$ and $\boldsymbol{V}$ can also be expressed as

$$\boldsymbol{U}=\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}(\boldsymbol{\sigma}\times\boldsymbol{L}) \tag{4.13}$$

and

$$\boldsymbol{V}=\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}(\boldsymbol{\sigma}\times\boldsymbol{L}). \tag{4.14}$$

Thus

$$\boldsymbol{U}^{\dagger}=(\boldsymbol{\sigma}\times\boldsymbol{L})\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} \tag{4.15}$$

and

$$\boldsymbol{V}^{\dagger}=(\boldsymbol{\sigma}\times\boldsymbol{L})\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}. \tag{4.16}$$

The operators R, P, $\boldsymbol{U}$, and $\boldsymbol{V}$ factorize functions of the commuting operators $\mathscr{H}$, $\boldsymbol{\sigma}\cdot\boldsymbol{L}$, $\boldsymbol{L}^2$, and $\tilde{J}_z$. For example,

$$R^{\dagger}(\pm\omega)R(\pm\omega)=2m_0(\mathscr{H}\pm\tfrac{1}{2}\omega\mp\omega\boldsymbol{\sigma}\cdot\boldsymbol{L}) \tag{4.17}$$

$$U_0^{\dagger}U_0=(\boldsymbol{\sigma}\times\boldsymbol{L})_0^2=\boldsymbol{L}^2-\tilde{J}_z^2+\tfrac{1}{4} \tag{4.18}$$

$$\begin{aligned}(U_{\pm1})^{\dagger}U_{\pm1}&=[(\boldsymbol{\sigma}\times\boldsymbol{L})_{\pm1}]^{\dagger}(\boldsymbol{\sigma}\times\boldsymbol{L})_{\pm1}\\&=(\boldsymbol{\sigma}\cdot\boldsymbol{L}\mp\tilde{J}_z)^2-\tfrac{1}{4}\end{aligned} \tag{4.19}$$

where in the last two calculations we have used (4.13). Similar results apply for $\boldsymbol{P}$ and $\boldsymbol{V}$.

The effect of $R(\omega)$ on $|njmq\rangle$ is the same as that of (4.3) with $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ replaced by its eigenvalues, that is, of

$$R_j(\omega)=-\mathrm{i}p_r+[1\pm\eta(j+\tfrac{1}{2})]r^{-1}+m_0\omega r \tag{4.20}$$

where the upper (lower) sign applies if R_j acts on $|njmq\rangle_{\mathrm{U}}$ ($|njmq\rangle_{\mathrm{L}}$). In the coordinate representation

$$p_r=-\mathrm{i}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) \tag{4.21}$$

and $R_j(\omega)$ are first-order differential operators. These differential operators are shift operators for the quantum numbers N and l in the radial coordinate-space wavefunctions F_{Nl} and G_{Nl} in (5.4); they are the shift operators obtained by applying the factorization method [15] to the differential form of $\mathscr{H}$ (see (4.17)). Similarly, in the coordinate representation one obtains first-order differential operators from $\boldsymbol{U}$ (see (4.2)); these are shift operators for j, m, and l in the spinor spherical harmonics $V^{\pm}_{jml}$ in (5.4).

Similar remarks apply to $P(\omega)$ and $\boldsymbol{V}$. In the momentum representation

$$r_p=\mathrm{i}\left(\frac{\partial}{\partial p}+\frac{1}{p}\right) \tag{4.22}$$

and thus from P and $\boldsymbol{V}$ one obtains first-order differential operators that are shift operators for the momentum-space wavefunctions.

By replacing $\mathscr{H}$ with its eigenvalues in the energy shift operators (3.29) one obtains operators that are linear in $\boldsymbol{p}$. The latter operators can also be derived by the factorization method [5], and in the coordinate representation they are first-order differential shift operators for N in the radial coordinate-space wavefunctions F_{Nl} and G_{Nl} in (5.4). Similarly, from (3.30) one obtains shift operators for the radial momentum-space wavefunctions.

5. Matrix elements, phase factors and wavefunctions

The matrix elements of $\boldsymbol{r}$ can be written down using (3.14), (3.15), (3.17), and (3.18). The results are given in table 1. By repeated use of these results one can evaluate the matrix elements of $x^{n_1}y^{n_2}z^{n_3}$ where n_i are non-negative integers.

Table 1. The non-zero matrix elements of $\boldsymbol{r}$. In these formulae $|njmq\rangle_{\mathrm{U}}$ and $|njmq\rangle_{\mathrm{L}}$ are the elements $|\phi\rangle$ and $|\chi\rangle$, respectively, in (2.15).

${}_{\mathrm{U}}\langle n\pm 1, j+\eta, m+\mu, q+\eta\|r_\mu\|njmq\rangle_{\mathrm{U}} = \pm d_+(1+m_0/E_{n\pm1,j+\eta})^{1/2}\alpha^{\pm}_{\mu}(njm)$
${}_{\mathrm{U}}\langle n\pm 1, j-\eta, m+\mu, q-\eta\|r_\mu\|njmq\rangle_{\mathrm{U}} = \pm d_+(1+m_0/E_{n\pm1,j-\eta})^{1/2}[\alpha^{\mp}_{-\mu}(n\pm 1, j-\eta, m+\mu)]^*$
${}_{\mathrm{L}}\langle n\pm 1, j-\eta, m+\mu, q+\eta\|r_\mu\|njmq\rangle_{\mathrm{L}} = \mp d_-(1-m_0/E_{n\pm1,j-\eta})^{1/2}\beta^{\pm}_{\mu}(njm)$
${}_{\mathrm{L}}\langle n\pm 1, j+\eta, m+\mu, q+\eta\|r_\mu\|njmq\rangle_{\mathrm{L}} = \mp d_-(1-m_0/E_{n\pm1,j+\eta})^{1/2}[\beta^{\mp}_{-\mu}(n\pm 1, j+\eta, m+\mu)]^*$
where
$d_\pm = \mathrm{i}[2(2m_0\omega)^{1/2}\{\eta(2j+1)\mp 1\}]^{-1}(1\pm m_0/E_{nj})^{1/2}$

The shift operations in section 3 contain the twelve phase factors $\theta^{\pm}_{\mu}(njm)$ and $\phi^{\pm}_{\mu}(njm)$. We now deduce relationships between these phase factors. From (3.27) and (3.28) we see that the choice of η for the phase factors in (3.30) and (3.31) requires

$$[\theta^{\mp}_{\mu}(n\pm 2, j, m)]^*\theta^{\pm}_{\mu}(njm) = \eta. \tag{5.1}$$

We can transform $|njmq\rangle_{\mathrm{U}}$ into $|n\pm 1, j+\eta, m+\mu, q+\eta\rangle_{\mathrm{L}}$ in two different ways, namely, by applying $D_\mu(\pm\omega)$ and then (2.16), or by applying (2.16) and then $D^{\dagger}_{\mu}(\pm\omega)$. Comparing the results we find

$$[\phi^{\mp}_{-\mu}(n\pm 1, j+\eta, m+\mu)]^* = -\theta^{\pm}_{\mu}(njm). \tag{5.2}$$

According to (5.1) and (5.2), only three of the twelve phase factors can be chosen independently. For example, if we choose

$$\theta^{+}_{0} = -\eta\mathrm{i} \qquad \theta^{+}_{\pm 1} = \pm\mathrm{i} \tag{5.3}$$

then (5.1) and (5.2) yield $\theta^{-}_{0} = -\mathrm{i}$, $\theta^{-}_{\pm 1} = \pm\eta\mathrm{i}$, $\phi^{+}_{0} = -\mathrm{i}$, $\phi^{+}_{\pm 1} = \mp\eta\mathrm{i}$, $\phi^{-}_{0} = -\eta\mathrm{i}$, and $\phi^{-}_{\pm 1} = \mp\mathrm{i}$.

With the choice (5.3) the representative of the Dirac ket (2.15) in the coordinate representation is, for $j = l\pm\frac{1}{2}$,

$$\Psi^{\pm}(\boldsymbol{r}, t) = \begin{pmatrix} \mathrm{i}F_{Nl}(r)\,V^{\pm}_{jml}(\hat{\boldsymbol{r}}) \\ G_{Nl}(r)\,V^{\mp}_{jml'}(\hat{\boldsymbol{r}}) \end{pmatrix} \exp(-\mathrm{i}Et). \tag{5.4}$$

Here

$$F_{Nl}(r)=\left[\frac{1}{2}\left(1+\frac{m_0}{E}\right)\right]^{1/2} C_{Nl}R_{Nl} \tag{5.5}$$

$$G_{Nl}(r)=(\text{sign } E)\left[\frac{1}{2}\left(1-\frac{m_0}{E}\right)\right]^{1/2} C_{N'l'}R_{N'l'} \tag{5.6}$$

where

$$R_{Nl}={}_1F_1(-N;\, l+\tfrac{3}{2};\, m_0\omega r^2)(m_0\omega r^2)^{l/2}\exp(-\tfrac{1}{2}m_0\omega r^2) \tag{5.7}$$

$$C_{Nl}=(m_0\omega)^{3/2}\eta^N\left[\frac{2^{l-N+2}(2N+2l+1)!!}{\pi^{1/2}N!\{(2l+1)!!\}^2}\right]^{1/2} \tag{5.8}$$

$l'=l-\eta$, $N'=N-\frac{1}{2}(1-\eta)$, N is the radial quantum number (see (3.33) and (3.34)), and E is given by (2.35) and (3.33). The angular functions in (5.4) are the spinor spherical harmonics

$$V^{\pm}_{jml}(\hat{\boldsymbol{r}})=(2l+1)^{-1/2}\begin{pmatrix}(l\pm m+\frac{1}{2})^{1/2}Y_{l,m-1/2}(\hat{\boldsymbol{r}})\\ \pm(l\mp m+\frac{1}{2})^{1/2}Y_{l,m+1/2}(\hat{\boldsymbol{r}})\end{pmatrix} \tag{5.9}$$

where Y_{lm} is a spherical harmonic defined as in [16]. In the above we have supposed l is a non-negative integer†. It is elementary to show that the radial momentum operator (4.4) is Hermitian with respect to the wavefunctions (5.4).

If in (5.4)-(5.9) we make the substitutions $r\to(m_0\omega)^{-1}p$, $\hat{\boldsymbol{r}}\to\hat{\boldsymbol{p}}$, and in (5.8) we replace $(m_0\omega)^{3/2}$ with‡ $(m_0\omega)^{-3/2}(-\eta\mathrm{i})^l$, we obtain the representative of the Dirac ket (2.15) in the momentum representation. The radial operator (4.9) is Hermitian with respect to these momentum-space wavefunctions.

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† The use of half-integral values of l in relativistic systems has been discussed in [17].

‡ The phase factor $(-\eta\mathrm{i})^l$ is required by the factor $-\mathrm{i}$ in (4.6).